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DISTANCE IN SPLITTING AND COSPLITTING GRAPHS

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ABSTRACT
The graph S(G) obtained from a graph G(V,E), by adding a new vertex w for every vertex v V and joining w to all neighbours of v in G, is called the splitting graph of G. The cosplitting graph CS(G) is obtained from G, by adding a new vertex wi for each vertex vi and joining wi¬ to all vertices of G which are not adjacent to vi in G. In this paper, we study the properties related to distance in splitting and cosplitting graphs.

INTRODUCTION
Throughout this paper, we consider only finite, simple and undirected graphs. For notations and terminology, we follow [4]. For any vertex $v \in V$ in a graph G(V,E), the *open neighbourhood* N(v) of v is the set of all vertices adjacent to v. That is, $N(v) = \{u \in V / uv \in E\}$. The *closed neighbourhood* N[v] of v is defined by $N[v] = N(v) \cup \{v\}$. A vertex of degree one is called a *pendant vertex*. A *full vertex* of a graph G is a vertex which is adjacent to all other vertices of G.
A subset S of V is called a *dominating set* of G if every vertex in V – S is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality taken over all dominating sets in G. A dominating set S of a connected graph G is said to be a *connected dominating set* of G if the induced sub graph <S> induced by S is connected. The minimum cardinality taken over all such connected dominating sets is the *connected domination number* and is denoted by $\gamma_c(G)$.

In a graph G, the *distance* d(u,v) between any two vertices u and v is the length of a shortest path between them. The *eccentricity* e(u) of a vertex u is the distance of a farthest vertex from u. The *radius* rad(G) of G is the minimum eccentricity and the *diameter* diam(G) of G is the maximum eccentricity of the graph G. A vertex v is called an *eccentric vertex* of a vertex u if d(u,v) = e(u). A vertex u with e(u) = rad(G) is called a *central vertex*. A graph G for which rad(G) = diam(G) is called a *self centered graph* of radius rad(G). Or equivalently a graph is *self centered* if all of its vertices are central vertices.

KM Kathiresan and Marimuthu [7] have introduced a new type of graph called radial graph. Two vertices of a graph G are said to be *radial* to each other if the distance between them is equal to the radius of the graph. The *Radial graph* R(G) of a graph G, is a graph with vertex set V(G) and two vertices in R(G) are adjacent if and only if they are radial in G. And if G is disconnected, then two vertices in R(G) are adjacent if they belong to different components of G. A graph G is called a *radial graph* if R(H) = G, for some graph H. For further details on radial graphs, one can refer [6] and [7].

The following theorem [2] gives the following characterisation for a graph to be radial which is needed for further studies in this paper.
Result A[2] A graph $G \ncong K_{m,n}$ is radial if and only if $\gamma_c(G) \neq 2$.
The concept of splitting graph of a graph was introduced by Sampath Kumar and Walikar [9]. For a graph G, the graph S(G), obtained from G, by adding a new vertex w for every vertex $v \in V$ and joining w to all vertices of G adjacent to v, is called the s*plitting graph* of G. This resembles the method of taking clone of every vertex in a graph. For example, a graph G and its splitting graph S(G) are shown in Figure 1.

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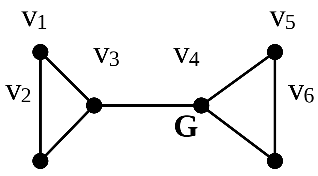

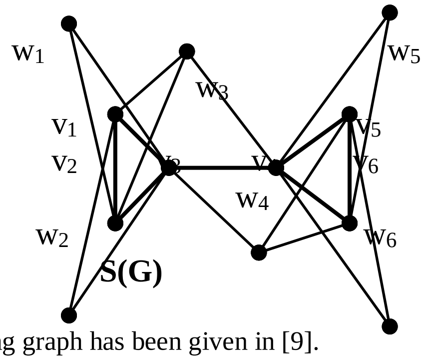

Figure 1

A necessary and sufficient condition for a graph to be a splitting graph has been given in [9].

Result B[9] A graph G is a splitting graph if and only if V(G) can be partitioned into two sets V_1 and V_2 such that there exists a bijective mapping f from V_1 to V_2 and $N(f(v)) = N(v) \cap V_1$, for any $v \in V_1$.

On a similar line, Ponraj and Somasundaram [8] have introduced the concept of degree splitting graph DS(G) of a graph G. For a graph G = (V, E) with vertex set partition $V_i = \{v \in V \,/\, d(v) = i\}$, the *degree splitting graph* DS(G) is obtained from G, by adding a new vertex w_i for each partition V_i that contains at least two vertices and joining w_i to each vertex of V_i.

For example, a graph G and its degree splitting graph DS(G) are shown in Figure 2.For further details on degree splitting graphs, one can refer [3] and [8].

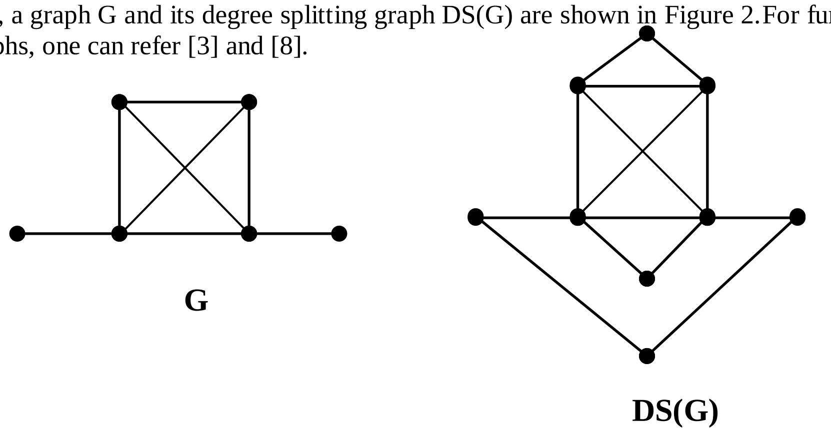

Figure 2

Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The *cosplitting graph* CS(G) is the graph obtained from G, by adding a new vertex w_i for each vertex v_i and joining w_i to all vertices which are not adjacent to v_i in G. For example, a graph G and its cosplitting graph CS(G) are shown in Figure 3.

Note that when we superimpose CS(G) on S(G), we get the graph $G \vee K_n^c$. In [1], the properties of cosplitting graphs have been studied and the graphs for which splitting and cosplitting graphs are isomorphic have been characterised.

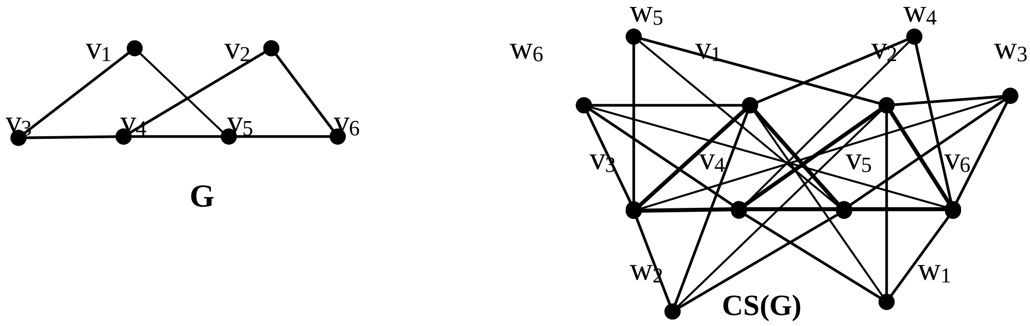

Figure 3

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In fact it has been proved that,

Result C[1] A graph G is a cosplitting graph if and only if V(G) can be partitioned into two sets V_1 and V_2 such that there exists a bijection f from V_1 to V_2 which satisfies the following conditions: (i) $N(v) \cup N(f(v)) = V \setminus f(N(v))$ and (ii) $N(v) \cap N(f(v)) = \phi$, for any $v \in V_1$.

In this paper, we compare the properties related to distance of a graph with its splitting and cosplitting graphs and obtain some interesting results on them.

DISTANCE IN SPLITTING GRAPHS

Throughout this paper, let $v_1, v_2, ..., v_n$ represent the vertices of any graph of order n. And let $u_1, u_2, ..., u_n$ be the corresponding newly added vertices in its splitting graph S(G). In this section, let us study the distance properties of splitting graphs.

The following facts can be easily verified:

Fact 1 For any graph G, $rad(S(G)) \geq 2$ and equality holds if and only if $rad(G) \leq 2$.

For, any splitting graph never contains a full vertex as $d(u_i, v_i) = 2$ for every i, $1 \leq i \leq n$.

Fact 2 $diam(S(K_{m,n})) = 3$ for all m, $n \geq 1$.

Fact 3 $diam(S(P_n)) = diam(P_n) = n - 1$ if and only if $n \geq 4$.

Fact 4 $diam(S(P_2)) = 3$ and $diam(S(P_3)) = 3$.

Fact 5 The eccentricity of any vertex v_i will not be affected in S(G). Hence $diam(G) \leq diam(S(G))$.

For, clearly for any newly added vertex u_i in S(G) corresponding to v_i in G, $d(v_i, v_j) = d(v_i, u_j)$ for all j, $1 \leq j \leq n$.

Proposition 6 $S(K_n)$ is a self centered graph of radius two for $n \geq 3$.

Proof It is easy to note that $S(K_n) \cong K_n \vee K_n^c$. Therefore $d(u_i, v_i) = 2$ in $S(K_n)$. Hence $e(v_i) = 2$ for any i, $1 \leq i \leq n$. Also $d(u_i, u_j) = 2$ implies $e(u_i) = 2$. Hence $S(K_n)$ is a self centered graph of radius two. ■

The following proposition characterises cycles for which splitting graphs are self centered.

Proposition 7 $S(C_n)$ is self centered if and only if $n \neq 4, 5$.

Proof It can be easily verified that $S(C_3)$ is a self centered graph of radius two. Let C_n be any cycle with $E(C_n) = \{v_nv_1, v_iv_{i+1} / 1 \leq i \leq n - 1\}$. In $S(C_n)$, $d(u_i, u_{i+1}) = 3$ for $1 \leq i \leq n - 1$ and $d(u,u_n) = 3$. Also $e(u_i) = e(v_i)$ if $e(v_i) > 2$. Hence for any $n \geq 6$, $S(C_n)$ is self centered. If we consider $S(C_4)$ or $S(C_5)$, $e(u_i) = 3$ and $e(v_i) = 2$, for any i, $1 \leq i \leq n$. Hence we can conclude that $S(C_4)$ and $S(C_5)$ are not self centered. ■

Theorem 8 If a graph G contains at least two full vertices, then S(G) is self centered of radius two.

Proof Let G be any graph with at least two full vertices. If $G \cong K_n$, then S(G) is self centered of radius two. Hence assume that G is not a complete graph. Every full vertex of G is of eccentricity two in S(G). It is obvious that $d(u_i, u_j) = d(u_i, v_j) = d(v_i, u_j) = d(v_i, v_j) = 2$ when $v_iv_j \notin E(G)$. If $v_iv_j \in E(G)$, then $d(u_i, u_j) = 2$ since every v_i and v_j have at least one common neighbour because of the presence of more than one full vertex in G. Hence S(G) is self centered of radius two. ■

Theorem 9 Let G be a graph with exactly one full vertex. Then S(G) is self centered if and only if $\delta(G) > 1$.

Proof Let G be any graph with exactly one full vertex say v_m. Suppose S(G) is self centered. Then rad(G) = 2. Now $e(v_m) = 2$ in S(G) since it is not adjacent to u_m. If possible, let $\delta(G) = 1$. Let v_t be a pendant vertex in G. Then $d(u_t, u_m) = 3$. This is a contradiction to the fact that S(G) is self centered of radius 2. Therefore $\delta(G) > 1$.

Conversely let $\delta(G) > 1$. Then every vertex v_i, $i \neq m$, is adjacent to at least one vertex v_j, $j \neq m$ in G. Then every two vertices in S(G) have a common neighbour and so $e(v) = 2$ for every vertex v in S(G). Hence S(G) is self centered. ■

It is very clear that C_4 and $C_3 \times P_2$ are self centered graphs, whereas $S(C_4)$ is not self centered but $S(C_3 \times P_2)$ is self centered. Hence we discuss the conditions under which splitting graphs are self centered.

Theorem 10 Let G be a self centered graph of radius two. Then S(G) is self centered of radius two if and only if every edge of G lies in a triangle.

Proof Let G be any self centered graph of radius two. Suppose that S(G) is also self centered. Since $e(v_i) = 2$ in G, $e(v_i) = 2$ in S(G) also. If possible, let v_iv_j be any edge in S(G) which does not lie in any triangle. Then $d(u_i, u_j) = 3$ since u_i and u_j have no common neighbour. This is a contradiction to the fact that S(G) is self centered of radius two. Hence every edge of G lies in a triangle.

Conversely suppose G is a self centered graph of radius two with every edge lying in a triangle. Then it is clear that every two vertices in S(G) have at least one common neighbour. Therefore S(G) is also a self centered graph of radius two. ■

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We have already seen in Fact 5 that diam(G) $\leq$ diam(S(G)). C_4 is an example for a graph with diam(C_4) < diam(S(C_4)). The following theorem gives a condition for diam(S(G)) = diam(G).

Theorem 11 For any graph G of diameter at least three, diam(S(G)) = diam(G).

Proof Let G be any graph with diameter at least three. Then G contains at least one vertex v_j such that $e(v_j) \geq 3$. Now we examine the eccentricities of vertices in S(G). By Fact 5, $e_{S(G)}(v_i) = e_G(v_i)$. Also $d(u_i,v_i) = 2$. Hence $e(u_i) \geq 2$. In addition we know that $N(u_i) = N(v_i) \cap V(G)$ (Result B) and hence $d(u_i, v_j) = d(v_i, v_j) = d(v_i, u_j)$. Therefore $d(u_i, u_j) \neq d(v_i, v_j)$ only when v_i and v_j are adjacent. In such a case, $d(u_i, u_j)$ is at most 3.

In particular, $d(u_i, u_j) = 2$ or 3 according as v_i, v_j have a common neighbour or no common neighbour. Or equivalently, $d(u_i , u_j) \neq 3$ only when every neighbour v_j of v_i is adjacent to another neighbour of v_i, that is, when $<N(v_i)>$ has no isolated vertex. Thus we can conclude that,

$$e(u_i) = \begin{cases} 2 & \text{if } e(v_i)<3 \text{ and } <N(v_i)> \text{has no isolated vertex} \\ 3 & \text{if } e(v_i)<3 \text{ and } <N(v_i)> \text{has isolated vertex} \\ e(v_i) & \text{if } e(v_i) \geq 3 \end{cases}.$$

Therefore for a graph with diameter greater than two, diam(S(G)) = diam(G). ■

Corollary 12 If G is a self centered graph of radius at least three, then S(G) is also self centered of radius, rad(G). ■

Corollary 13 For any graph G, diam(S(G)) > diam(G) if and only if diam(G) $\leq$ 2 and G contains a vertex v_i such that $<N(v_i)>$ has an isolated vertex.

DISTANCE IN COSPLITTING GRAPHS

For the vertices $v_1, v_2, \ldots, v_n$ of a graph G, let $u_1, u_2, \ldots, u_n$ be the corresponding newly added vertices in its cosplitting graph CS(G). In this section, let us study the properties related to distances in cosplitting graphs.

Fact 14 For any graph G, rad(CS(G)) $\geq$ 2.

For, since Δ(CS(G)) = |V(G)| = n, CS(G) does not contain a full vertex.

Fact 15 diam(CS($K_{m,n}$)) = 3 for all m, n $\geq$ 1 and diam(CS(K_n)) = 3 for all n $\geq$ 2.

Fact 16 The eccentricity of any vertex v_i is 2 in CS(G).

For, clearly any vertex in CS(G) is adjacent to either v_i or its corresponding newly added vertex w_i in CS(G). It is obvious that if CS(G) is self centered, then rad(CS(G)) = 2.

Fact 17 Since every vertex in G is adjacent to either v_i or w_i in CS(G), $d(w_i, w_j) \leq 3$, in a cosplitting graph. Hence diam(CS(G)) $\leq$ 3, for any graph G.

The following proposition gives a characterisation for paths having cosplitting graphs as self centered.

Proposition 18 CS(P_n) is a self centered graph of radius two if and only if n $\neq$ 2, 3 or 4.

Proof When n = 1, CS(P_1) $\cong$ K_2 which is self centered. When n = 2, CS(P_2) $\cong$ P_4 and $d(w_1, w_2) = 3$. When n = 3, $d(w_1, w_2) = 3$ in CS(P_3). When n = 4, $d(w_2, w_3) = 3$ in CS(P_4). But by Fact 16, for every i, $1 \leq i \leq n$, $e(v_i) = 2$ in CS(G). Hence CS(P_n) is not self centered when n = 2, 3, 4. When n $\geq$ 5, it can be easily verified that, for every i, $1 \leq i \leq n$, $e(v_i) = e(w_i) = 2$. Therefore CS(P_n) is self centered when n $\geq$ 5. ■

The following proposition characterises cycles for which cosplitting graphs are self centered.

Proposition 19 CS(C_n) is self centered if and only if n $\neq$ 3,4.

Proof Clearly CS(C_3) $\cong$ $C_3 \circ K_1$ which is not self centered as distance between any two pendant vertices is three. Also in CS(C_4), $d(w_i, w_j) = 3$, for any $i \neq j$, $1\leq i, j \leq n$, implying that CS(C_4) is not self centered. When n > 4, it can be easily verified that distance between any two vertices in CS(G) is two and hence it is a self centered graph of radius two. ■

Theorem 20 If a graph G contains at least one full vertex, then CS(G) is not self centered.

Proof Let G be any graph with at least one full vertex, say v_1. Then w_1 is a pendant vertex in CS(G). Also any w_i, $i \neq 1$, is not adjacent to v_1. Hence $d(w_1,w_3) = 3$, for all $i \neq 1$, $1 \leq i \leq n$. By Fact 16, for every i, $1 \leq i \leq n$, $e(v_i) = 2$ in CS(G). Therefore CS(G) is not self centered. ■

Note that the converse of above theorem is not true. For example, CS(C_4) is not self centered but C_4 has no full vertex.

Theorem 21 Let G($\ncong K_{m,n}$) be a graph with no full vertex. Then CS(G) is self centered if and only if G is a radial graph.

Proof Let G be any graph other than $K_{m,n}$ with $\Delta(G) < n - 1$. Suppose CS(G) is not self centered. Then there exist at least two vertices in CS(G), say w_1 and w_2 such that $d(w_1, w_2) = 3$. That is, $N(w_1) \cap N(w_2) = \phi$. Or equivalently, $N(v_1)^c \cap N(v_2)^c = \phi$ and hence $N(v_1) \cup N(v_2) = V(G)$. This means that there exist two adjacent

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vertices v_1 and v_2 in G such that $N(v_1) \cup N(v_2) = V(G)$. Therefore $\gamma_c = 2$ and also given that $G \ncong K_{m,n}$. Hence G is not radial (Result A), which is a contradiction. Conversely assume that G is not a radial graph. Retracing the same steps, we have CS(G) is not self centered. Therefore CS(G) is self centered if and only if G is a radial graph.

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