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ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $(x^2+y^2) - xy + x + y + 1 = 16z^2$

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ABSTRACT
The ternary homogeneous quadratic equation given by $(x^2+y^2)-xy+x+y+1=16z^2$ representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramided numbers are presented.

INTRODUCTION
The Diophantine equations offer an unlimited field for research due to their variety [1-3].In particular, one may refer [4-24] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $(x^2+y^2)-xy+x+y+1=16z^2$ representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS USED
1. Polygonal number of rank 'n' with sides m
$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$
2. Pronic number of rank 'n'
$$\Pr_n=n(n+1)$$

METHOD OF ANALYSIS
The ternary quadratic diophantine equation under consideration is
$$(x^2+y^2)-xy+x+y+1=16z^2 \quad (1)$$

The substitution of the linear transformations
$$x = u+v \;;\; y = u-v \qquad (u\neq 0, v\neq 0) \quad (2)$$
in (1) gives
$$(u+1)^2+3v^2=16z^2 \quad (3)$$
Taking u+1=U
in (3), it gives
$$U^2+3v^2=16z^2 \quad (4)$$
Now, (4) is solved through different methods and thus, different patterns of solutions to (1) are obtained.
Method: 1
Write (4) as
$$U^2-16z^2=-3v^2$$
i.e) $(U+4z)(U-4z)=-3v^2$ (5)
Choice (i)
Write (5) in the form of ratio as

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$$\frac{-3v}{(U-4z)}=\frac{(U+4z)}{v}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (6)$$

This is equivalent to the following two equations

$$\left.\begin{aligned}-\alpha U-3\beta v+4\alpha z=0\\ U\beta-\alpha v+4\beta z=0\end{aligned}\right\} \qquad (7)$$

Applying the method of cross multiplication, the above system of equations is satisfied by

$$U= u+1=4\alpha^2-12\beta^2$$
$$v=8\alpha\beta$$
$$Z= \alpha^2+3\beta^2 \qquad (8)$$

Substituting the values of u and v in (2), we get

$$x=4\alpha^2-12\beta^2-1+8\alpha\beta$$
$$y=4\alpha^2-12\beta^2-1-8\alpha\beta \qquad (9)$$

Thus (8) and (9) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(a,a)-y(a,a)-16t_{4,a}=0$

(2) $x(2a,a)+y(2a,a)-8t_{4,a}+2=0$

(3) $z(5b,5b)$ is a perfect square.

(4) $z(b,4b)$ is a perfect square.

(5) $x(1,a)+y(1,a)+z(1,a)+21t_{4,a}-7=0$

Choice (ii)

(5) is written in the form of ratio as

$$\frac{-3v}{(U+4z)}=\frac{(U-4z)}{v}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (10)$$

This is equivalent to the following two equations

$$\left.\begin{aligned}-\alpha U-3\beta v-4\alpha z=0\\ \beta U-\alpha v-4\beta z=0\end{aligned}\right\} \qquad (11)$$

Applying the method of cross multiplication, the above system of equations is satisfied by

$$U= u+1=-4\alpha^2+12\beta^2$$
$$v=-8\alpha\beta$$
$$z= \alpha^2+3\beta^2 \qquad (12)$$

Substituting the values of u and v in (2), we get

$$x=-4\alpha^2+12\beta^2-1-8\alpha\beta$$
$$y=-4\alpha^2+12\beta^2-1+8\alpha\beta \qquad (13)$$

Thus (12) and (13) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(2,a)+y(2,a)+z(2,a)-27t_{4,a}+30=0$

(2) $x(b,1)+y(b,1)+8t_{4,b}-22=0$

(3) $z(4b,4b)$ is a perfect square.

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(4) $y(b,2)-z(b,2)+21t_{4,b}-16pr_b-35=0$.

(5) $x(1,a)-z(1,a)-17t_{4,a}+8pr_b+6=0$

Choice (iii)

Also, (5) is written in the form of ratio as

$$\frac{U+4z}{-3v}=\frac{v}{U-4z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{14}$$

This is equivalent to the following two equations

$$\left.\begin{aligned} U\beta+3\alpha v+4\beta z=0 \\ -U\alpha+\beta v+4\alpha z=0 \end{aligned}\right\} \tag{15}$$

Applying the method of cross multiplication, the above system of equations is satisfied by

U= $u+1=12\alpha^2-4\beta^2$

$v=-8\alpha\beta$

z = $3\alpha^2+\beta^2$ (16)

Substituting the values of u and v in (2), we get

$x=12\alpha^2-4\beta^2-1-8\alpha\beta$

$y=12\alpha^2-4\beta^2-1+8\alpha\beta$ (17)

Thus (16) and (17) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(1,-a)-y(1,-a)-z(1,-a)+17t_{4,a}-16pr_a+3=0$

(2) $x(2,b)+y(2,b)+z(2,b)+7t_{4,b}-106=0$

(3) $x(2,-b)-z(2,-b)+21t_{4,b}-16pr_b-35=0$

(4) $x(a,a)+y(a,a)+z(a,a)-20t_{4,a}+2=0$

(5) $x(-a,-a)-y(-a,-a)-z(-a,-a)+20t_{4,a}=0$

Choice (iv)

Again (5) is written in the form of ratio as

$$\frac{U-4z}{-3v}=\frac{v}{U+4z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{18}$$

This is equivalent to the following two equations

$U\beta+3\alpha v-4\beta z=0$

$-U\alpha+\beta v-4\alpha z=0$ (19)

Applying the method of cross multiplication, the above system of equations is satisfied by

U= $u+1=-12\alpha^2+4\beta^2$

$v=8\alpha\beta$

z = $3\alpha^2+\beta^2$ (20)

Substituting the values of u and v in (2), we get

$x=-12\alpha^2+4\beta^2-1+8\alpha\beta$

$y=-12\alpha^2+4\beta^2-1-8\alpha\beta$ (21)

Thus (20) and (21) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(a,3)+y(a,3)+24t_{4,a}-70=0$

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(2) $x(-a,-a)-z(-a,-a)+4t_{4,a}+1=0$

(3) $x(a,a)+y(a,a)+16t_{4,a}+2=0$

(4) $x(1,-b)+y(1,-b)+z(1,-b)-9t_{4,b}+23=0$

(5) $x(3,a)+y(3,a)+z(3,a)-9t_{4,a}+191=0$

Method: 2

Assume $z=z(a,b)=a^2+3b^2$ (22)

where a,b are non-zero integers,

Write 16 as $16=(2+i2\sqrt{3})(2-i2\sqrt{3})$ (23)

Substituting (22) and (23) in (4) and employing the method of factorization, we've

$$(2U+i2\sqrt{3}v)(2U-i2\sqrt{3}v)=(2+i2\sqrt{3})(2-i2\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2$$

The above equation is equivalent to the system of equation

$$(2U+i2\sqrt{3}v)=(2+i2\sqrt{3})(a+i\sqrt{3}b)^2 \quad (24)$$

$$(2U-i2\sqrt{3}v)=(2-i2\sqrt{3})(a-i\sqrt{3}b)^2 \quad (25)$$

Equating the real and imaginary parts in (24)

U=u+1= $u(a,b)=2a^2-6b^2-12ab$

v= $v(a,b)=2a^2-6b^2+4ab$

Substituting the values of the u and v in (2), we've

x = $x(a,b)=4a^2-12b^2-8ab-1$ (26)

y = $y(a,b)=-16ab-1$ (27)

Thus (26) (27) and (25) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

(1) $x(-a,2)-y(-a,2)-20t_{4,a}+16pr_a+48=0$

(2) $y(a,1)+z(a,1)-17t_{4,a}+16pr_a-2=0$

(3) $(8b,8b)$ is a perfect square.

(4) $(12b,12b)$ is a perfect square.

(5) $-x(1,a)+y(1,a)+z(1,a)-23t_{4,a}+8pr_a+3=0$

Case(ii)

Write 16 as

$$16=(-2+i2\sqrt{3})(-2-i2\sqrt{3}) \quad (28)$$

Substituting (22) (28) in (4) and employing the method of factorization, we've

$$(-2U+i2\sqrt{3}v)(-2U-i2\sqrt{3}v)=(-2+i2\sqrt{3})(-2-i2\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2$$

The above equation is equivalent to the system of equation

$$(-2U+i2\sqrt{3}v)=(-2+i2\sqrt{3})(a+i\sqrt{3}b)^2 \quad (29)$$

$$(-2U-i2\sqrt{3}v)=(-2-i2\sqrt{3})(a-i\sqrt{3}b)^2 \quad (30)$$

Equating the real and imaginary parts in (29)

U=u+1= $u(a,b)=-2a^2+6b^2-12ab$

v= $v(a,b)=2a^2-6b^2-4ab$

Substituting the values of the u and v in (2), we've

x = $x(a,b)=-16ab-1$ (31)

y = $y(a,b)=-4a^2+12b^2-8ab-1$ (32)

Thus (31) (32) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

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Properties:-

(1) $x(-a,-a)+y(-a,-a)+16t_{4,a}+2=0$

(2) $x(1,a)+z(1,a)-19t_{4,a}+16pr_a=0$

(3) $(32b,32b)$ is a perfect square.

(4) $(24b,24b)$ is a perfect square.

(5) $x(1,a)+y(1,a)+z(1,a)+39t_{4,a}+24pr_a+5=0$

CONCLUSION

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic diophantine equation represented by $(x^2+y^2)-xy+x+y+1=16z^2$. As quadratic equations are rich in variety, one may search for other choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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