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International Journal OF Engineering Sciences & Management Research

CONSTRUCTION OF DIOPHANTINE QUADRUPLES WITH PROPERTY D(A PERFECT SQUARE)

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Keywords: Diophantine quadruples, System of equations.

MSC 2000 Mathematics Subject Classification: 11D99

ABSTRACT
This paper concerns with the study of construction of Diophantine quadruples such that the product of any two elements of the set added by a perfect square is a perfect square.

INTRODUCTION

Let q be a non-zero number. A set $\{a_1, a_2, ...a_m\}$ of non-zero rational is called a $D(q)$ - mtuple, if $a_i a_j + q$ is a square for all $1 \le i \le j \le m$. The mathematician Diophantus of Alexandria considered a variety of problems on indeterminant equations with rational or integers solutions. In particular, one of the problems was to find the sets of distinct positive rational numbers such that the product of any two numbers is one less than a rational square [14] and Diophantus found four positive rationals $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$ [4,5]. The first set of four positive integers with the same property, the set $\{1,3,8,120\}$ was found by Fermat. It was proved in 1969 by Baker and Davenport [3] that a fifth positive integer cannot be added to this set and one may refer [6, 7,11] for generalization. However, Euler discovered that a fifth rational number can be added to give the following rational Diophantine quintuple $\left\{1,3,8,120,\frac{777480}{8288641}\right\}$. Rational sextuples with two equal elements have been given in [2]. In this 1999, Gibs [13] found several examples of rational Diophantine sextuples, eg., $\left\{\frac{11}{192},\frac{35}{192},\frac{155}{27},\frac{512}{27},\frac{1235}{48},\frac{180873}{16}\right\}$, $\left\{\frac{17}{448},\frac{265}{448},\frac{2145}{448},252,\frac{23460}{7},\frac{2352}{7921}\right\}$.

All known Diophantine quadruples are regular and it has been conjectured that there are no irregular Diophantine quadruples [1,13] (this is known to be true for polynomials with integer co-efficients [8]). If so then there are no Diophantine quintuples. However there are infinitely many irregular rational Diophantine quadruples. The smallest is $\frac{1}{4},5,\frac{33}{4},\frac{105}{4}$. Many of these irregular quadruples are examples of another common type for which two of the subtriples are regular i.e., $\{a,b,c,d\}$ is an irregular rational Diophantine quadruple, while $\{a,b,c\}$ and $\{a,b,\mathrm{d}\}$ are regular Diophantine triples. These are known as semi – regular rational Diophantine quadruples. These are only finitely many of these for any given common denominator l and they can readily found.

Moreover in [12], it has been proved that $D(\mp k^2)$ - triple $\{k^2, k^2 \pm 1, 4k^2 \pm 1\}$ cannot be extended to a $D(\mp k^2)$ - quintuple. In [10], it has been proved that $D(-k^2)$ - triple $\{1, k^2+1, k^2+4\}$ cannot be extended to a $D(-k^2)$ - quadruple if $k \ge 5$.

 ISSN 2349-6193
Impact Factor (PIF): 2.243

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METHOD OF ANALYSIS

SECTION I:

In this section we search the diophantine quadruple (a,b,c,d) such that product of any two of them added with n^2k^2 is a perfect square. Also,the fourth tuple is either integer or rational number.

Consider $a=(6-n)k-6, \& b=(6+n)k-6$

Note that $ab+n^2k^2$ is a perfect square.

Let c be any non-zero integer such that

$$ac+n^2k^2=\alpha^2 \quad (1)$$

$$bc+n^2k^2=\beta^2 \quad (2)$$

From (1), we have

$$c=\frac{\alpha^2-n^2k^2}{a} \quad (3)$$

Assume $\alpha = X+((6-n)k-6)T$ (4)

$$\beta = X+((6+n)k-6)T \quad (5)$$

On substituting the value of (3) in (2) and by using (4) and (5), we get

$$X^2=\left[((6-n)k-6)((6+n)k-6)\right]T^2+n^2k^2$$

whose initial solution is $T_0=1,\ X_0=6(k-1)$

Thus $\alpha=6(k-1)+(6-n)k-6$

$$\beta=6(k-1)+(6+n)k-6$$

Therefore from (3)

$$c=24(k-1)$$

Let d be any non-zero integer such that

$$ad+n^2k^2=A^2 \quad (6)$$

$$bd+n^2k^2=B^2 \quad (7)$$

$$cd+n^2k^2=C^2 \quad (8)$$

Solving (6), (7) and (8) we get the value of d

$$d=\frac{24(k-1)}{n^2k^2}\left[144(k-1)^2-n^2k^2\right]$$

Substituting the value of d in (6),(7) & (8) then

$$ad+n^2k^2=\left[\frac{n^2k^2-144(k-1)^2+12nk(k-1)}{nk}\right]^2$$

$$bd+n^2k^2=\left[\frac{n^2k^2-144(k-1)^2-12nk(k-1)}{nk}\right]^2$$

$$cd+n^2k^2=\left[\frac{n^2k^2-288(k-1)^2}{nk}\right]^2$$

Therefore (a,b,c,d) is a mixed diophantine quadruple with property $D(n^2k^2)$ as the fourth tuple may not always be integer. In what follows, a few examples of diophantine quadruple integer are presented.

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Table.1

n	k	(a,b,c,d)
1	2	(4,8,24,840)
	3	(9,15,48,3024)
	4	(14,22,72,5760)
	6	(24,36,120,11880)
	8	(34,50,168,18354)
	12	(54,78,264,31680)
2	2	(2,10,24,192)
	3	(6,18,48,720)
	4	(10,26,72,1386)
	6	(18,42,120,2880)
	12	(42,90,264,7722)
3	2	(0,12,24,72)
	4	(6,30,72,576)
	8	(18,66,168,1890)
4	2	(-2,14,24,30)
	3	(0,24,48,144)

Section II:

In this section we search the diophantine quadruple (a,b,c,d) such that product of any two of them added with k^2 is a perfect square.

Assume $a = n^2$ and $b = n^2 2^{2n} + k.2^{n+1}$

$ab + k^2$ is a perfect square.

Let c be any non-zero integer such that

$$ac + k^2 = \alpha^2 \tag{9}$$

$$bc + k^2 = \beta^2 \tag{10}$$

From (9), we have

$$c = \frac{\alpha^2 - k^2}{a} \tag{11}$$

Assume $\alpha = X + n^2 T$ (12)

$$\beta = X + (n^2 2^{2n} + k.2^{n+1})T \tag{13}$$

On substituting the value of (11) in (10) and by using (12) and (13), we get

$$X^2 = \left[n^2 (n^2 2^{2n} + k.2^{n+1}) \right] T^2 + k^2$$

Whose initial solution is $T_0 = 1,\ X_0 = (n^2 2^n + k)^2$

Thus $\alpha = (n^2 2^n + k) + n^2$

$$\beta = (n^2 2^n + k) + (n^2 2^{2n} + k.2^{n+1})$$

Therefore from (11)

$$c = n^2 (2^n + 1)^2 + 2k(2^n + 1)$$

Let d be any non-zero integer such that

$$ad + k^2 = A^2 \tag{14}$$

$$bd + k^2 = B^2 \tag{15}$$

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$$cd + k^2 = C^2 \qquad (16)$$

Solving (14), (15) and (16) we get the value of d

$$d = \frac{1}{k^2}\left[20k^2 n^2 2^n (2^n+1) + 4k^2(n^2+k) + 8k^3 2^n + 4n^6 2^{2n}(2^n+1)^2 + 8kn^4(3.2^{2n} + 2.2^{3n} + 2^n)\right]$$

Substituting the value of d in (14),(15) & (16) then

$$ad + k^2 = \left[\frac{k^2 + 2n^2 2^n (n^2 2^n + n^2) + 2kn^2(2.2^n + 1)}{k}\right]^2$$

$$bd + k^2 = \left[\frac{k^2(4.2^n + 1) + 2k2^n(3n^2 2^n + 2n^2) + 2n^4 2^{2n}(2^n+1)}{k}\right]^2$$

$$cd + k^2 = \left[\frac{k^2(4.2^n + 3) + 2kn^2(3.2^{2n} + 4.2^n + 1) + 2n^4 2^n (2^n+1)^2}{k}\right]^2$$

Therefore (a,b,c,d) is a mixed diophantine quadruple with property $D(k^2)$ as the fourth tuple may not always be integer, a few numerical examples of diophantine quadruple integer are presented in the following table.

Table .2

n	k	(a,b,c,d)
1	1	(1,8,15,528)
	2	(1,12,21,320)
	3	(1,16,27,280)
	4	(1,20,33,273)
	6	(1,28,45,288)
2	1	(4,72,110,127092)
	2	(4,80,120,38808)
	4	(4,96,140,13920)
	5	(4,104,150,10500)
	8	(4,128,180,6384)
	10	(4,144,200,5304)
3	1	(9,592,747,15922760)
	2	(9,608,765,4188844)
	3	(9,624,783,1957200)
	4	(9,640,801,1156340)
	6	(9,672,837,565500)
	8	(9,704,873,348880)
	9	(9,720,891,288360)
4	1	(16,4128,4658,1230623940)
	2	(16,4160,4692,312317256)

Section III:

In this section we search the diophantine quadruple (a,b,c,d) such that product of any two of them added with 4.2^{2n} is a perfect square.

Assume $a = Carl_n = 2^{2n} - 2^{n+1} - 1$ and $b = Ky_n = 2^{2n} + 2^{n+1} - 1$

$ab + 4.2^{2n}$ is a perfect square.

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Let c be any non-zero integer such that

$$ac+4.2^{2n}=\alpha^2 \tag{17}$$

$$bc+4.2^{2n}=\beta^2 \tag{18}$$

From (17), we have

$$c=\frac{\alpha^2-4.2^{2n}}{a} \tag{19}$$

Assume $\alpha=X+(2^{2n}-2^{n+1}-1)T$ (20)

$$\beta=X+(2^{2n}+2^{n+1}-1)T \tag{21}$$

On substituting the value of (19) in (18) and by using (20) and (21), we get

$$X^2=\left[(2^{2n}-2^{n+1}-1)(2^{2n}+2^{n+1}-1)\right]T^2+4.2^{2n}$$

whose initial solution is $T_0=1,\ X_0=(2^{2n}-1)^2$

Thus
$$\alpha=2.2^{2n}-2^{n+1}-2$$
$$\beta=2.2^{2n}+2^{n+1}-2$$

Therefore from (19)

$$c=4(2^{2n}-1)$$

Let d be any non-zero integer such that

$$ad+k^2=A^2 \tag{22}$$

$$bd+k^2=B^2 \tag{23}$$

$$cd+k^2=C^2 \tag{24}$$

Solving (22), (23) and (24) we get the value of d

$$d=4.2^{4n}-16.2^{2n}+16-\frac{4}{2^{2n}}$$

Substituting the value of d in (22),(23) & (24) then

$$ad+4.2^{2n}=\left[2.2^{3n}-2.2^{2n}-6.2^n+2+\frac{2}{2^n}\right]^2$$

$$bd+4.2^{2n}=\left[2.2^{3n}+2.2^{2n}-6.2^n+2-\frac{2}{2^n}\right]^2$$

$$cd+4.2^{2n}=\left[4.2^{3n}-10.2^n+\frac{4}{2^n}\right]^2$$

Remark:

It is seen that the fourth tuple d is integer only when $n=1$ and the corresponding quadruple is (-1,7,12,15) with the property $D(4^2)$.

CONCLUSION

To conclude one may construct a Diophantine quadruples with suitable properties.

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