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THE SPACES OF GENERALISED STRONGLY ALMOST CONVERGENT SEQUENCES

Ajaya Kumar Singh*
*Department of Mathematics, P.N.College (Auto), Khordha

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ABSTRACT
The object of the present paper is to introduce almost convergence by means of generalised binomial coefficients and definenew sequence spaces and various inclusions and topological properties.

STRONGLY ALMOST CONVERGENT SEQUENCES

Let $T_{k,n}(x_p)$ be defined by

$$T_{k,n}(x_p) = \begin{cases} x_{n+p}, & k = 0 \\ \dfrac{1}{A_n^{k,\delta}} \displaystyle\sum_{v=p}^{p+n} A_{p+n-v}^{k-1,\delta}\, x_v, & k > 0 \end{cases}$$

A sequence $x = \{x_n\}$is said to be strongly almost convergent to a number s if

$$\frac{1}{n}\sum_{i=1}^{n} \left|T_{k-1,i}\left(x_{p_j}\right) - s\right| \to 0 \; as \; n \to \infty \; uniformly \; in \; p.$$

$$\text{where } Q_k(x) = \inf_{p_1,\dots,p_r} \; \lim_{n\to\infty} \sup \frac{1}{r}\sum_{j=1}^{r}\left|T_{k,n}\left(x_{p_j}\right)\right| \tag{1}$$

We define$[f_k]_q$, for $q \geq 1$ and $k \geq 1$the spaces of strongly f_k- almost convergent sequences with index q :

$$[f_k]_q = \left\{x: \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \left|T_{k-1,i}(x_p) - s\right|^q = 0, \; uniformly \; in \; p\right\}. \tag{2}$$

We observe that

$$\lim_{n\to\infty} \left\{\frac{1}{n+1}\sum_{i=0}^{n} \left|T_{k-1,i}(x_p) - s\right|^q\right\}^{1/q} = \sup_i \left|T_{k-1,i}(x_p) - s\right| \tag{3}$$

This implies that $T_{k-1,i}(x_p) \to s$as $i \to \infty$, uniformly in p. it follows immediately, by Cauchy's Theorem on limit, that

$$\frac{1}{n+1}\sum_{i=0}^{n} T_{k-1,i}(x_p) \to s \quad as \; n \to \infty, uniformly \; in \; p.$$

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Thus the space f_{k-1} may be regarded as the case $q = \infty$ of space $[f_k]_q$. It is also easy to show

byHÖlder's inequality, that if (2) is true for some q, it holds for any smaller q.

Further for any s, we have

$$\left\{\frac{1}{n+1}\sum_{i=0}^{n} \left|T_{k-1,i}(x_p\ ) - s\right|^{q'}\right\}^{1/q'} \leq \left\{\frac{1}{n+1}\sum_{i=0}^{n} \left|T_{k-1,i}(x_p\ ) - s\right|^{q}\right\}^{1/q}$$
$$\leq \sup_i \left\{\frac{1}{n+1}\left|T_{k-1,i}(x_p\ ) - s\right|\ \right\} \qquad (4)$$

whenever$0 < q' < q < \infty,\ k > 0.$
It is worth mentioning that (as may easily be proved by (3) and (4)),

$$[f_k]_\infty \subset [f_k]_q \subset [f_k]_{q'}$$

PRELIMINARY RESULTS

First we note that the coefficients $A_n^{\alpha,\delta}$ are defined by the following power series:

$$\frac{1}{(1-x)^{\alpha+1}}\left(\log\frac{a}{1-x}\right)^{\delta} = \sum_{n=0}^{\infty} A_n^{\alpha,\delta}\, x^n, a > 2$$

From this it follows that

$$\sum_{n=0}^{n} A_v^{\alpha,\delta}\, A_{n-v}^{\alpha',\delta'} = A_n^{\alpha+\alpha'+1,\delta+\delta'} \qquad (5)$$

Before proving our results, we first prove the following lemmas.

Lemma 1 suppose that $k' > k > 0$. Then

$$T_{k',n}(x_p) = \frac{1}{A_n^{k',\delta}}\sum_{l=0}^{n} A_{n-l}^{k'-k-1,0}\, A_l^{k,\delta}\, T_{k,l}(x_p) \qquad (6)$$

Proof we see that (6) is equal to

$$\frac{1}{A_n^{k',\delta}}\sum_{l=0}^{n} A_{n-l}^{k'-k-1,0}\, A_l^{k,\delta} \sum_{i=0}^{l} A_{l-i}^{k-1,\delta}\, x_{i+p}\ . \qquad (7)$$

Inverting the order of summation, the expression (6) becomes using special case of (5)

$$\frac{1}{A_n^{k',\delta}}\sum_{i=o}^{n} x_{i+p} \sum_{l=i}^{n} A_{n-l}^{k'-k-1,0} A_{l-i}^{k-1,\delta} = \frac{1}{A_n^{k',\delta}}\sum_{i=0}^{\infty} x_{i+p}.A_{n-i}^{k'-1,\delta} = T_{k',n}(x_p)$$

This proves lemma.

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Lemma 2 Suppose that $t_n = \frac{s_0+s_1+\cdots+s_n}{n+1}$, where $s_n = a_0 + a_1 + \cdots + a_n$

If $\lim_{n\to\infty} s_n$ exists, then $\lim_{n\to\infty} t_n$ exists and has the same value.

This lemma is well known.

SOME RESULTS

Theorem 1 Suppose that $k' > k > 0$. Then

$$f_k \subset f_{k'}$$

Proof From (6), we have

$$T_{k',n}(x_p) = \frac{1}{A_n^{k'}} \sum_{l=0}^{n} A_{n-l}^{k'-k-1,0} A_l^{k,\delta} T_{k,l}(x_p) \tag{9}$$

It follows that

$$\overline{\lim_{n\to\infty}} |T_{k',n}(x_p)| \le \overline{\lim_{n\to\infty}} |T_{k,n}(x_p)| \tag{10}$$

We deduce from (10) that $k' > k > 0$ and

$$\overline{\lim_{n\to\infty}} T_{k,n}(x_p) = 0, \text{ then } \overline{\lim_{n\to\infty}} T_{k',n}(x_p) = 0$$

Consequently for $k' > k > 0$

$$\overline{\lim_{n\to\infty}} T_{k',n}(x_p) \quad exists\ uniformly\ in\ p.$$

$$\text{whenever } \overline{\lim_{n\to\infty}} T_{k,n}(x_p) \quad exists\ uniformly\ in\ p.$$

In other words $f_k \subset f_{k'}$.

This proves Theorem 1.

Theorem 2 $[f_k]_q \subset f_k \quad and\ [f_k]_q - \lim x = s \Rightarrow f_k - \lim x = s.$

Proof To prove theorem, we observe that if $x \in [f_k]_q$ for any q such that

$1 \le q \le +\infty, then\ x \in [f_k]$. We may suppose that $q = 1$, i.e.

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} |T_{k-1,i}(x_p) - s| = 0, \qquad uniformly\ in\ p, k \ge 1.$$

Now the result for $q = 1$ follows from the following inequality:

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$$\left|\sum_{i=o}^{n} T_{k-1,i}(x_p\ \ ) - s\right| \leq \sum_{i=0}^{n} \quad |T_{k-1,i}(x_p\ \ ) - s|$$

Now suppose that $q > 1$

We have from Lemma 1

$$T_{k\ \ ,n}(x_p) = \frac{1}{A_n^k} \sum_{l=0}^{n} A_l^{k-1}\, T_{k-1,l}(x_p\ \ ) - s$$

And so

$$|T_{k,n}(x_p\ \ ) - s| \leq \frac{1}{A_n^k} \sum_{l=0}^{n} A_l^{k-1}\, |T_{k-1,l}(x_p\ \ ) - s|$$

Now applying HÖlder's inequality we have

$$|T_{k,n}(x_p\ \ ) - s| \leq \frac{1}{A_n^k} \left(\sum_{l=0}^{n} |T_{k-1,l} - s|^p\right)^{1/p} \left(\sum_{l=0}^{n} A_l^{(k-1)p'}\right)^{1/p'}$$

$$= o(n^{-k})\, o\left(n^{\frac{1}{p}}\right)\, o\left(n^{\frac{p'(k-1)+1}{p'}}\right) = o\left(n^{-k+\frac{1}{p}+k-1+\frac{1}{p'}}\right)$$

$$= o(n^{-k+k}) \qquad = o(1)$$

Hence $|T_{k-1,l}(x_p\ \ ) - s| = o(1)$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=o}^{n} T_{k,i}(x_p\ \ ) = s \text{ uniformly in } \ p, k \geq 0 \text{ (by Lemma 2)}$$

Hence $x \in f_k$

This proves Theorem 2.

Theorem 3 if $q > 1, k' > k > \frac{1}{q}$ and $k \geq 0$,then$[f_{k+1}]_q \subset f_{k'}$.

Proof We may evidently suppose that $s = 0$. Let$x \in [f_{k+1}]_q$. By Lemma 1, we have

$$|T_{k',n}(x_p\ \ )| \leq \frac{1}{A_n^{k'}} \sum_{l=0}^{n} A_{n-l}^{k'-k-1}\, A_l^k |T_{k,n}(x_p)|.$$

Applying HÖlder's inequality with indices $q$and q', we get

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$$|T_{k',n}(x_p)| \leq \frac{1}{A_n^{k'}}\left\{\sum_{l=0}^{n} A_l^k |T_{k,l}(x_p)|^q\right\}^{1/q}\left\{\sum_{l=0}^{n} A_l^k\left(A_{n-l}^{k'-k-1}\right)^{q'}\right\}^{1/q'}$$

$$\leq \frac{1}{A_n^{k'}}\left(A_n^k\right)^{\frac{1}{q}}\sum_{l=0}^{k}|T_{k,l}(x_p)|^q o(1)\left(\sum_{l=0}^{n} A^k (n-l)^{q'(k'-k-1)}\right)^{1/q'},$$

$$= o(1)n^{-k'}.n^{\frac{k}{q}}.n^{\frac{1}{q}}o(1).o(1)\left(n^{k+q'(k'-k-1)+1}\right)^{\frac{1}{q'}}, \text{using } k' > k > \frac{1}{q}$$

$$= o(1)n^{-k'+\frac{k}{q}+\frac{1}{q}+k'-k-1+\frac{1}{q'}}$$

$$= o(1)n^{\frac{k}{q}+\frac{k}{q'}-k}\left(\text{using}\frac{1}{q}+\frac{1}{q'}=1\right)$$

$$= o(1)n^{k-k} \qquad = o(1)$$

Hence $T_{k',n}(x_p) = o(1)$ as $n \to \infty$ uniformly in p.

Applying Lemma 2, we see that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=o}^{n} T_{k',i}(x_p) = o(1) \quad as\ n \to \infty\ uniformly\ in\ p.$$

Hence $x \in f_{k'}$.

This proves Theorem 3.

We generalize the definition of the space of M-convergent sequences defined by Maddox [3], and we also generalize the Theorem 1 of Maddox [3] and Theorem 2 of Das and Mishra [2].

DEFINITION

Let d be any sublinear functional on l_∞. We write $\{l_\infty, d\}$ to be the set of all linear functional η_k on l_∞ such that $\eta_k > d$, that is $\eta_k(x) \leq d(x)$, for all $x \in l_\infty$. We now define an M_k-limit on l_∞ to be a linear functional η_k on l_∞, such that

$$\eta_k(x) \leq Q_k(x) \qquad for\ all\ x \in l_\infty$$

where $Q_k(x)$ is defined by (1).

Since $\eta_k(x) \leq Q_k(x)$ every Banach limit is also an M_k-limit.

It is natural to define $x \in l_\infty$ to be M_k $-$convergent to s if and only if

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$\eta_k(x-s)=0 \quad for\ all\ M_k$-limits $\eta_k(11)$

Let $[M_k]$ denote the space of all M_k-convergent sequences.

Theorem 4 If $x \in l_\infty$ then

Proof If (2) holds, then for each $\epsilon > 0$ there exists r such that

$$\lim\sup_n \frac{1}{r}\sum_{j=1}^{r}\left|T_{k-1,n}(x_j\quad)-s\right| < \epsilon,$$

Hence $Q_k(x-s) \le \epsilon$. Now if M_k is any M_k-limit then $M_k(y) \le Q_k(y)$on l_∞, and

$$-M_k(y)=M_k(-y) \le Q_k(-y)=Q_k(y), so\ |M_k(y)| \le Q_k(y).$$

Hence $|M_k(x-s)| \le \epsilon$, which implies $x \in [M_k]$

Since every Banach limits is also on M_k- limit the inclusion $[M_k] \subset f_k$ is immediate.

This completes the proof.

Theorem 5 $[f_k]=[M_k]$

Proof In view of the inclusions $[f_k] \subset [M_k] \subset f_k$, it is enough to show that $[M_k] \subset [f_k]$.

Definition for $x \in l_\infty$,

$$I_k(x)=\overline{\lim_r}\quad \sup_n \frac{1}{r}\sum_{j=1}^{r}\left|T_{k-1,n}(x_j\quad)\right|$$

Then by corollary of theorem 1, proved by Das and Mishra [2], writing $|x|=|(x_n)|_{n\ge 0}$ in

place of $x=(x_n)$, we obtain $Q_k(x)=I_k(x)$.

Now, let $x\ \epsilon\ [M_k]$, so there exists a (real) such that

$$\eta_k(x-s)=0\ \ for\ all\ \eta_k(x)\ \epsilon\ \{l_\infty, I_k\}.$$

By the Hahn-Banach Theorem, there exists $\eta_0(x)\ \epsilon\ \{l_\infty, I_k\}$

such that

$$\eta_0(x-s)=I_k(x-s).$$

Hence, $I_k(x-s)=0$,
which implies that $x\ \epsilon\ [f_k]$.

This completes the proof.

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